ANALOG OF PRANDTL-MEYER FLOW IN A MAGNETIC FIELD PARALLEL TO THE VELOCITY FIELD

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Prandtl-Meyer magnetogasdynamic flows were investigated by M. N. Kogan [1, 2], who found the regions of existence of these flows and, by using the method of characteristics to solve the equation of motion of the gas, indicated the typical features of flows of this type.

The vortical problem of the plane flow of ideal gas with infinite conductivity in the case of the adiabatic process for a magnetic field parallel to the velocity field is considered. An analog of Sedov's equation [3] in the plane of the variables "pressure, stream function," the criterion of existence and the equation of generalized Prandtl-Meyer flow are obtained. A solution for the latter equation in the transonic region is found. A method of constructing such flows for the case of a compressible fluid flowing past certain profiles is described.

1. The magnetohydrodynamic equations for an ideal gas with infinite conductivity in steady-state flow have the form [4]

$$\frac{1}{2}\nabla \mathbf{v}^2 + \operatorname{rot}\mathbf{v}\times\mathbf{v} = -\frac{1}{\rho}\nabla p - \frac{1}{4\pi\rho}\frac{1}{4\pi\rho}\mathbf{H}\times\operatorname{rot}\mathbf{H}$$
$$\mathbf{H}\operatorname{div}\mathbf{v} - (\mathbf{H}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{H} = 0,$$
$$\operatorname{div}\rho\,\mathbf{v} = 0, \quad p/\rho^\gamma = \mathrm{const} \quad (\gamma = c_p/c_v). \tag{1.1}$$

Equations (1.1) still have the same form when we convert to the dimensionless quantities v_{12}, H_1, p_1, and ρ_1, defined by the relationships [5]

$$\mathbf{v} = v_0\cdot\mathbf{v}_1, \quad \mathbf{H} = H_0\cdot\mathbf{H}_1, \quad \rho = \rho_0\rho_1, \quad p = p_0 p_1, \quad \mathbf{r} = l\mathbf{r}_1,$$
$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y, \qquad H_0 = v_0\sqrt{\rho_0}, \qquad p_0 = \rho_0 v_0^2.$$

Here ρ_0 and v_0 are the density and velocity of the incoming undisturbed flow from infinity and l is a characteristic length. Henceforth we will regard Eqs. (1.1) as dimensionless equations in which the subscripts for the relative quantities are omitted.

It was shown in [6] that in the case of motion of a gas in a magnetic field parallel to the velocity of the flow the equality

$$\mathbf{H} = k\rho\mathbf{v} \qquad (k = \mathrm{const}) \tag{1.2}$$

holds. In this case the Bernoulli integral and the velocity vortex are expressed in the form

$$\frac{v^2}{2} + \frac{\gamma}{\gamma-1}\rho^{\gamma-1} = \frac{\gamma}{\gamma-1},$$
$$\Omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \frac{k^2}{4\pi}\left(\frac{\partial\rho v_y}{\partial x} - \frac{\partial\rho v_x}{\partial y}\right). \tag{1.3}$$

It is clear from (1.3.2) that a magnetic field parallel to the velocity field causes vortical motion in the case of the adiabatic process (1.1.4), governed by the system [5]

$$v_x = \lambda(\rho)\frac{\partial\varphi}{\partial x} = \frac{1}{\rho}\frac{\partial\psi}{\partial y}, \qquad v_y = \lambda(\rho)\frac{\partial\varphi}{\partial y} = -\frac{1}{\rho}\frac{\partial\psi}{\partial x},$$
$$\frac{v^2}{2} + \frac{\gamma}{\gamma-1}\rho^{\gamma-1} = \frac{\gamma}{\gamma-1}, \qquad \lambda(\rho) = \left[1 - \frac{k^2\rho}{4\pi}\right]^{-1}. \tag{1.4}$$

Here $\varphi = \varphi(x, y)$ is the virtual potential and $\psi = \psi(x, y)$ is the stream function.

2. As independent variables we take the stream function ψ and the pressure p (Sedov variables [3], p. 300), completing the conversion from the formulas

$$\frac{\partial\psi}{\partial x} = \frac{\partial y/\partial p}{\Delta}, \qquad \frac{\partial\psi}{\partial y} = -\frac{\partial x/\partial p}{\Delta},$$
$$\Delta = \frac{\partial x}{\partial\psi}\frac{\partial y}{\partial p} - \frac{\partial x}{\partial p}\frac{\partial y}{\partial\psi} \neq 0, \quad \frac{\partial v_x}{\partial y} = -\frac{D(v_x, x/\psi, p)}{\Delta},$$
$$\frac{\partial v_y}{\partial x} = \frac{D(v_y, y/\psi, p)}{\Delta}. \tag{2.1}$$

Hence, after simple transformations we arrive at the expressions

$$\frac{\partial x}{\partial\psi} = A(p,\theta), \qquad A(p,\theta) = \frac{\partial}{\partial p}(v\sin\theta) + v\mu\frac{\cos 2\theta}{\sin\theta},$$
$$\frac{\partial y}{\partial\psi} = -B(p,\theta), \qquad B(p,\theta) = \frac{\partial}{\partial p}(v\cos\theta) - v\mu\frac{\cos 2\theta}{\cos\theta},$$
$$\mu = \mu(\rho) = \frac{k^2}{a^2(4\pi - k^2\rho)},$$
$$v_x = v\cos\theta, \qquad v_y = v\sin\theta. \tag{2.2}$$

Here v is the velocity modulus, θ is the angle of inclination of the velocity vector to the x-axis, and a is the speed of sound. Using formulas (2.2), we obtain the relationships

$$\frac{\partial x}{\partial p} = -\frac{1}{\partial\theta/\partial\psi}\left(\cos^2\theta\frac{\partial B}{\partial p} + \sin\theta\cos\theta\frac{\partial A}{\partial p}\right),$$
$$\frac{\partial y}{\partial p} = -\frac{1}{\partial\theta/\partial\psi}\left(\sin\theta\cos\theta\frac{\partial B}{\partial p} + \sin^2\theta\frac{\partial A}{\partial p}\right). \tag{2.3}$$

Introducing the complex variable z = x + iy, we obtain from (2.2) and (2.3) the condition of solvability in the form

$$\frac{1}{v}\frac{\partial}{\partial p}\left(v^2\frac{\partial\theta}{\partial p}\right) + 2\frac{\partial}{\partial p}[v\mu\operatorname{ctg}2\theta] =$$
$$= \frac{\partial}{\partial\psi}\left\{\frac{1}{\partial\theta/\partial\psi}\left[v\left(\frac{\partial\theta}{\partial p}\right)^2 + 2v\mu\operatorname{ctg}2\theta\frac{\partial\theta}{\partial p} - \frac{\partial^2 v}{\partial p^2}\right]\right\}. \tag{2.4}$$

Equation (2.4) is a nonlinear equation and is a generalization of the Sedov equation to the case of magnetogasdynamics.

Function $z(\psi, p)$ is determined by means of quadratures from Eqs. (2.2) and (2.3), which are equivalent to the differential relationship

$$dz = e^{i\theta}\left\{\left[v\frac{\partial\theta}{\partial p} + 2v\mu\operatorname{ctg}2\theta - i\frac{\partial v}{\partial p}\right]d\psi -\right.$$
$$\left. - \frac{1}{\partial\theta/\partial\psi}\left(\cos\theta\frac{\partial B}{\partial p} - \sin\theta\frac{\partial A}{\partial p}\right)dp\right\}. \tag{2.5}$$

For a prescribed relationship $\rho = \rho(p)$ functions $\mu(\rho)$ and $v(\psi, p)$ will be known, and Eqs. (2.4) and (2.5) will be a complete system of equations of the problem.

3. In the derivation of Eq. (2.4) we made the essential assumption that $\partial\theta/\partial\psi \neq 0$. If $\partial\theta/\partial\psi \equiv 0$, then we obtain a flow of liquid for which the angle θ depends only on the pressure: $\theta = \theta(p)$. In [3] (p. 318) it was shown that a Prandtl-Meyer flow corresponds to such a condition in the case of isentropic flow of an ideal gas. The general solution of Eq. (2.2) in this case will be

$$z(\psi, p) = -\psi\left[i\frac{\partial}{\partial p}(ve^{i\theta}) - 2\mu v\operatorname{ctg}2\theta e^{i\theta}\right] + \omega(p), \tag{3.1}$$

where $\omega(p)$ is an arbitrary function of its argument.

To determine the function $\theta = \theta(p)$ we can easily find the equation

$$\left(\frac{\partial\theta}{\partial p}\right)^2 + 2\mu\operatorname{ctg}2\theta\frac{\partial\theta}{\partial p} - \frac{1}{v}\frac{\partial^2 v}{\partial p^2} = 0. \tag{3.2}$$

Hence we obtain

$$\frac{\partial\theta}{\partial p} = -\mu\operatorname{ctg}2\theta \pm \left(\mu^2\operatorname{ctg}^2 2\theta + \frac{1}{v}\frac{\partial^2 v}{\partial p^2}\right)^{1/2}. \tag{3.3}$$

Equation (3.3) can provide a qualitative characteristic of the analog of Prandtl-Meyer flows in magnetogasdynamics. From (3.3) we obtain the criterion of existence of flows of this type:

$$\mu^2\operatorname{ctg}^2 2\theta \geqslant -\frac{1}{v}\frac{\partial^2 v}{\partial p^2}. \tag{3.4}$$

It is clear that first, in the presence of a magnetic field parallel to the velocity field, the flow analogous to Prandtl-Meyer flows will be vortical in the case of the adiabatic process and, secondly, such a flow can exist not only in the supersonic, but also in the subsonic, regions.

From (3.3), using a Newtonian binomial expansion, we arrive at the equation

$$\frac{d\theta}{\partial p} = -\mu \operatorname{ctg} 2\theta \pm \mu \operatorname{ctg} 2\theta \left[1 + \frac{1}{2\mu^2 v \operatorname{ctg}^2 2\theta} \frac{\partial^2 v}{\partial p^2} + \ldots\right]. \quad (3.5)$$

We consider the case of transonic generalized Prandtl-Meyer flow. Then, since equalities $\partial^2 v/\partial p^2 = 0$ and $M = 1$ are equivalent, we confine ourselves to two terms. In accordance with the plus or minus sign before the root in (3.3) we obtain two families of solutions of the equations

$$\frac{d\theta}{\partial p} = \frac{1}{2\mu v} \frac{\partial^2 v}{\partial p^2} \operatorname{tg} 2\theta,$$

$$\frac{d\theta}{dp} = -\frac{1}{2\mu v} \frac{\partial^2 v}{\partial p^2} \operatorname{tg} 2\theta - 2\mu \operatorname{ctg} 2\theta. \quad (3.6)$$

The solution of Eq. (3.6.1) will be

$$\theta = \frac{1}{2} \arcsin \left[c_1 \exp\left(\int \frac{1}{\mu v} \frac{\partial^2 v}{\partial p^2} dp\right)\right]. \quad (3.7)$$

Equation (3.6.2) can be put in the form

$$\frac{1}{2} \frac{d}{dp} \ln \sin 2\theta = -2\mu \left(\frac{1}{\sin^2 2\theta} - 1\right) - \frac{1}{2\mu v} \frac{\partial^2 v}{\partial p^2}. \quad (3.8)$$

The substitution $\sin^2 2\theta = t$ brings (3.3) to the linear inhomogeneous equation

$$t' = Rt - 8\mu \qquad \left(R = 8\mu - \frac{2}{\mu v} \frac{\partial^2 v}{\partial p^2}\right). \quad (3.9)$$

Then the solution of (3.6.2) will be

$$\theta = \frac{1}{2} \arcsin \left\{\exp\left(4\int R\, dp\right) \times \right.$$

$$\left. \times \int \left[\mu \exp\left(-2\int R\, dp\right)\right] dp + c_2\right\}^{1/2}. \quad (3.10)$$

It follows from (3.7) and (3.10) that the point $\theta = 0$ is unattainable in the given formulation of the problem. The special nature of this point can also be seen from Eq. (3.2).

In the absence of a magnetic field ($k = 0$, $\mu = 0$) and taking into account that the conversion is made from $\partial^2 v/\partial p^2 < 0$ to $\partial^2 v/\partial p^2 > 0$, we find that $\theta = 0$ when $M = 1$ [7], for which we must put $c_2 = 0$, and restrict c_1 sufficiently.

The sign in (3.3) and, hence, in the solution from (3.7) and (3.10) is chosen so that the flow turns through an increasingly greater angle with increase in the local Mach number.

4. Using the arbitrariness of function $\omega(p)$ in solution (3.1), we can construct an analog of the Prandtl-Meyer flow with an arbitrary streamline. In this way in some cases we can construct the flow around given profiles which are not finite bodies and ensure the presence of a zone of undisturbed flow (for instance, a convex wall, where the flow is undisturbed at the start of the bend).

Let it be required to find a solution for flow around a body whose contour can be considered as prescribed by the streamline $\psi = 0$; the equation of this line in complex form can be put in the form

$$z = F(\theta), \quad (4.1)$$

where as a parameter we take the slope of the tangent to the x-axis which is obviously equal to θ. Replacing θ by p from solution (3.7) or (3.10), we find

$$z = F(\theta(p)) = F(p). \quad (4.2)$$

Comparing (3.1) and (4.2) for $\psi = 0$, we obtain

$$z = F(p) = \omega(p). \quad (4.3)$$

Thus, the arbitrary function $\omega(p)$ is completely determined when the contour of the streamlined body is prescribed.

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